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VIBRATION ANALYSIS OF A ROTATING TRUNCATED CIRCULAR CONICAL SHELL

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Abstract—In this paper, a method is presented to study the free vibrations of a rotating truncated circular conical shell with simply-supported boundary conditions. The method is based on the use of Love's first approximation theory and it includes the effects of initial hoop tension and the centrifugal and coriolis accelerations. Results are obtained for the frequency characteristics at different modes and various geometric properties, the effects of cone angle on the frequency characteristics are also discussed. To validate the present analysis, comparisons are made with a very long rotating cylindrical shell and a non-rotating truncated circular conical shell and very good agreement is obtained. © 1997 Elsevier Science Ltd.

INTRODUCTION

Shell structures are increasingly being used in many industries and as a consequence, the vibration of cylindrical shell structures has been extensively studied. This has been extended to studies on the vibration of rotating cylindrical shell as there are also engineering applications of a rotating shell in industry, for example, in the drive shafts of gas turbines, motors and rotor systems.

Bryan (1890) studied a rotating cylinder using the analysis of a spinning ring and discovered the travelling-mode phenomenon. The effects of coriolis acceleration on the free vibration of an infinitely long rotating cylindrical shell were investigated using a ring mode by Di Taranto and Lessen (1964); Srinivasan and Lauterbach (1971) also studied a similar problem. Works on rotating composite cylindrical shells have been carried out by Rand and Stavsky (1991) and Chun and Bert (1993). A finite element analysis for rotating shells has been undertaken by Chen *et al.* (1993). Recently extensive works on the vibration of cylindrical shell, both stationary and rotating, have been carried out by the first author. The effects of boundary conditions on the frequency characteristics for a multi-layered cylindrical shell using beam functions were studied by Lam and Loy (1994a). Analysis of rotating laminated cylindrical shells using different thin shell theories have also been carried out by Lam and Loy (1995a). Studies have also been carried out on rotating laminated composite (Lam and Loy, 1994b) and sandwich-type cylindrical shells (Lam and Loy, 1995b).

A natural progression to studies on the vibration of cylindrical shell structures is to extend it to conical shell structures. This has in fact been done on vibration of stationary conical shells, however extensive search of the literature has thus far shown that no work on rotating conical shell has been carried out.

Bacon and Bert (1967) used Rayleigh–Ritz method to study free vibration of both isotropic and orthotropic conical shells. The effects of transverse shear deformation have also been included in a similar study by Kayran and Vinson (1990). Sivadas and Ganesan (1991, 1992) used a semi-analytical finite element method to study vibration of laminated conical shells of varying thickness. Recently, Tong (1993a, 1993b) has carried out studies on the free vibration of isotropic, orthotropic and composite laminated conical shells.

Noting the lack of published works on the vibration of rotating conical shells, the present paper presents a method based on Love's first approximation theory to study the

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free vibration of a rotating truncated circular conical shell with simply-supported boundary conditions. The method is an extension of the first author's previous works, however as can be seen later, the amount of analysis required to extend the study from rotating cylindrical shell to rotating conical shell is quite considerable. Results are presented on the frequency characteristics of rotating truncated circular conical shell, the influences of various geometrical properties on the frequency characteristics are also considered in the present paper. The present formulations are also validated against results available in the literature for the vibration of an infinitely long rotating cylindrical shell and a stationary conical shell and found to be accurate.

GOVERNING EQUATIONS AND NUMERICAL IMPLEMENTATION

Consider a truncated circular conical shell rotating about its symmetrical and horizontal axis at an angular velocity Ω as shown in Fig. 1. In the figure, α is the cone angle, L the length, h the thickness and a and b are the radii at the two ends, respectively. The reference surface of the conical shell is taken to be at its middle surface where an orthogonal co-ordinate system (x, y, z) is fixed and r is a radius at co-ordinate point (x, y, z). The deformations of the rotating conical shell in the x, y and z directions are defined by u, v, w, respectively.

Chen *et al.* (1993) obtained the general equations for the vibration of rotating shells of revolution by using the linear approximation method. In their paper, the fundamental equilibrium equations were established by vector derivations and then asymptotically expanded into two groups of equations corresponding, respectively, to the basic and additional states. The former group of equations is referred to the equilibrium equations for the centrifugal forces. The latter group is the equation of motion involving the coriolis forces. The superimposition of these two state equations yields the general governing equations for the vibration of rotating shells of revolution.

Based on these general equations (see Chen *et al.*, 1993), transforming their curvilinear co-ordinate system into the present orthogonal co-ordinate system and then imposing the geometric characteristics of the rotating conical shell on the equations, the governing equations of motion for a truncated circular rotating conical shell can be derived directly as

$$\frac{\partial N_x}{\partial x}+\frac{1}{r}\frac{\partial N_{x\theta}}{\partial \theta}+\frac{N_\theta^0}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2}-r\cos\alpha\frac{\partial w}{\partial x}\right)+\frac{\sin\alpha}{r}(N_x-N_\theta)+2\rho h\Omega\sin\alpha\frac{\partial v}{\partial t}-\rho h\frac{\partial^2 u}{\partial t^2}=0, \qquad (1)$$

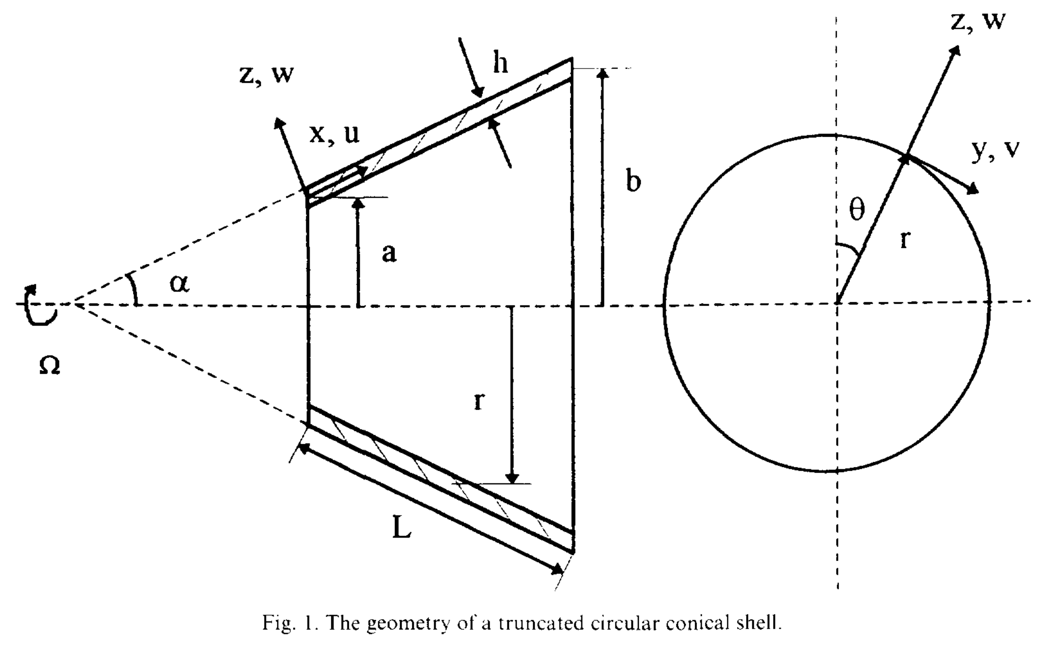

Fig. 1. The geometry of a truncated circular conical shell.

$$\frac{\partial N_{x\theta}}{\partial x}+\frac{1}{r}\frac{\partial N_\theta}{\partial \theta}+\frac{\cos\alpha}{r}\frac{\partial M_{x\theta}}{\partial x}+\frac{\cos\alpha}{r^2}\frac{\partial M_\theta}{\partial \theta}+\frac{N_\theta^0}{r^2}\left(r\frac{\partial^2 u}{\partial x\partial\theta}+\sin\alpha\frac{\partial u}{\partial\theta}+r\sin\alpha\frac{\partial v}{\partial x}\right)$$

$$+2\frac{\sin\alpha}{r}N_{x\theta}-2\rho h\Omega\left(\sin\alpha\frac{\partial u}{\partial t}+\cos\alpha\frac{\partial w}{\partial t}\right)-\rho h\frac{\partial^2 v}{\partial t^2}=0, \quad (2)$$

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{2}{r}\frac{\partial^2 M_{x\theta}}{\partial\theta\partial x}+\frac{1}{r^2}\frac{\partial^2 M_\theta}{\partial\theta^2}+\frac{2\sin\alpha}{r}\frac{\partial M_x}{\partial x}-\frac{\sin\alpha}{r}\frac{\partial M_\theta}{\partial x}+\frac{N_\theta^0}{r^2}\left(\frac{\partial^2 w}{\partial\theta^2}-r\cos\alpha\frac{\partial u}{\partial x}\right)$$

$$+\frac{N_\theta^0}{r^2}(w\cos^2\alpha+u\sin\alpha\cos\alpha)-\frac{\cos\alpha}{r}N_\theta+2\rho h\Omega\cos\alpha\frac{\partial v}{\partial t}-\rho h\frac{\partial^2 w}{\partial t^2}=0 \quad (3)$$

here,

$$r=r(x)=a+x\sin\alpha \quad (4)$$

$$N_\theta^0=\rho h\Omega^2 r^2=\rho h\Omega^2(a+x\sin\alpha)^2 \quad (5)$$

where ρ is the density of the conical shell, N_θ^0 is defined as the initial hoop tension because of the centrifugal force effect; N_i and M_i are the force and moment resultants, respectively, and can be represented by

$$(N_x, N_\theta, N_{x\theta})=\int_{-h/2}^{h/2}(\sigma_x,\sigma_\theta,\sigma_{x\theta})\,dz$$

$$(M_x, M_\theta, M_{x\theta})=\int_{-h/2}^{h/2}(\sigma_x,\sigma_\theta,\sigma_{x\theta})z\,dz. \quad (6)$$

The geometric relations of deformations for the reference surface of the rotating conical shell can be written as

$$e_1=\frac{\partial u}{\partial x}$$

$$e_2=\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u\sin\alpha+w\cos\alpha}{r}$$

$$e_{12}=\frac{1}{r}\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial x}-\frac{v\sin\alpha}{r}$$

$$\kappa_1=-\frac{\partial^2 w}{\partial x^2}$$

$$\kappa_2=-\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}+\frac{\cos\alpha}{r^2}\frac{\partial v}{\partial\theta}-\frac{\sin\alpha}{r}\frac{\partial w}{\partial x}$$

$$\kappa_{12}=2\left(-\frac{1}{r}\frac{\partial^2 w}{\partial x\partial\theta}+\frac{\sin\alpha}{r^2}\frac{\partial w}{\partial\theta}+\frac{\cos\alpha}{r}\frac{\partial v}{\partial x}-\frac{v\sin\alpha\cos\alpha}{r^2}\right) \quad (7)$$

here e_1 and e_2 are the strains of the reference surface in the meridional direction and in the circumferential direction, respectively, e_{12} is the shear strain of the reference surface and κ_1, κ_2 and κ_{12} are the reference surface curvatures, respectively.

Based on Love's first approximation theory (1952), the strain component at the coordinate point (x, y, z) may be defined by

$$e_x = e_1 + z\kappa_1$$
$$e_\theta = e_2 + z\kappa_2$$
$$e_{x\theta} = e_{12} + z\kappa_{12} \quad (8)$$

where e_x and e_θ are the strains in the meridional and circumferential directions, respectively, and $e_{x\theta}$ is the shear strain at a distance z from the reference surface.

For the isotropic conical shell, the constitutive relations may be represented by

$$\begin{Bmatrix} N_x \\ N_\theta \\ N_{x\theta} \\ M_x \\ M_\theta \\ M_{x\theta} \end{Bmatrix} = \frac{Eh}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 & 0 & 0 & 0 \\ \mu & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\mu}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{h^2}{12} & \frac{\mu h^2}{12} & 0 \\ 0 & 0 & 0 & \frac{\mu h^2}{12} & \frac{h^2}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-\mu)h^2}{12} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_{12} \\ \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{Bmatrix} \quad (9)$$

where E and μ are the modulus of elasticity and Poisson's ratio, respectively.

For the truncated circular rotating conical shell, the simply-supported boundary conditions at both ends are given by

$$v = 0, \quad w = 0, \quad N_x = 0, \quad M_x = 0 \quad \text{at } x = 0, L. \quad (10)$$

The displacement field may be taken as

$$u = U\cos\left(\frac{m\pi x}{L}\right)\cos(n\theta+\omega t)$$
$$v = V\sin\left(\frac{m\pi x}{L}\right)\sin(n\theta+\omega t)$$
$$w = W\sin\left(\frac{m\pi x}{L}\right)\cos(n\theta+\omega t) \quad (11)$$

where $\omega(\text{rad s}^{-1})$ is the natural circular frequency of the present conical shell and n is an integer representing the circumferential wave number of the shell.

It is obvious that the trial function (11) can accurately satisfy the geometric boundary conditions and approximately satisfy the force boundary conditions, but a comparison of the calculating results we obtain with those available in the open literatures concerning rotating cylindrical shell and non-rotating conical shell shows good agreement, as shown in the following Table 1 and Table 2. And furthermore, for such complicated partial differential equations with variable coefficients (1)–(3) as we discuss in this paper, the trial function (11) is obviously simpler and thus more applicable.

Substituting equation (7) into equation (9) and then substituting the resulting expression into equations (1)–(3), the governing equations obtained are a set of partial differential equations with variable coefficients; in other words, the coefficients of the differential operator in the equations are functions of the coordinate variable x. The simplified expressions of the resulting equations are given as follows:

Table 1. Comparison of the frequency parameter $f = \omega^* = \omega b\sqrt{((1-\mu^2)\rho/E)}$ for the free vibration of a very long rotating cylindrical shell by taking $\alpha = 0$ in the present formulations ($m = 1, \mu = 0.3, R = a = b = 1, h/b = 0.002$)

Ω (rps)	n	Chen *et al.* (1993) ω_b^*	Chen *et al.* (1993) ω_f^*	Present results ω_b^*	Present results ω_f^*
0.05	2	0.00167	0.00142	0.00170	0.00145
	3	0.00448	0.00429	0.00450	0.00431
	4	0.00848	0.00833	0.00850	0.00835
	5	0.01370	0.01353	0.01367	0.01355
0.1	2	0.00180	0.00130	0.00189	0.00139
	3	0.00457	0.00419	0.00465	0.00428
	4	0.00855	0.00826	0.00863	0.00834
	5	0.01371	0.01347	0.01379	0.01355

Subscripts b and f denote the backward and forward waves, respectively. From equation (45) of Chen *et al.* (1993):

$$\omega_b^* = \frac{2n}{n^2+1}\Omega + \sqrt{\frac{n^2(n^2-1)^2}{n^2+1}\frac{Eh^2}{\rho(1-\mu^2)12R^2} + \frac{n^4+3}{(n^2+1)^2}\Omega^2}$$

$$\omega_f^* = \frac{2n}{n^2+1}\Omega - \sqrt{\frac{n^2(n^2-1)^2}{n^2+1}\frac{Eh^2}{\rho(1-\mu^2)12R^2} + \frac{n^4+3}{(n^2+1)^2}\Omega^2}.$$

Table 2. Comparison of the frequency parameter $f = \omega b\sqrt{((1-\mu^2)\rho/E)}$ for the free vibration of a non-rotating truncated circular conical shell with simply-supported boundary condition ($m = 1$, $\mu = 0.3$, $h/b = 0.001$, $L\sin\alpha/b = 0.25$)

n	$\alpha = 30°$ Present	$\alpha = 30°$ Irie (1984)	$\alpha = 45°$ Present	$\alpha = 45°$ Irie (1984)	$\alpha = 60°$ Present	$\alpha = 60°$ Irie (1984)
2	0.8420	0.7910	0.7655	0.6879	0.6348	0.5722
3	0.7376	0.7284	0.7212	0.6973	0.6238	0.6001
4	0.6362	0.6352	0.6739	0.6664	0.6145	0.6054
5	0.5528	0.5531	0.6323	0.6304	0.6111	0.6077
6	0.4950	0.4949	0.6035	0.6032	0.6171	0.6159
7	0.4661	0.4653	0.5921	0.5918	0.6350	0.6343
8	0.4660	0.4654	0.6001	0.5992	0.6660	0.6650
9	0.4916	0.4892	0.6273	0.6257	0.7101	0.7084

$$L_{11}u + L_{12}v + L_{13}w = 0 \tag{12}$$

$$L_{21}u + L_{22}v + L_{23}w = 0 \tag{13}$$

$$L_{31}u + L_{32}v + L_{33}w = 0 \tag{14}$$

where $L_{ij}(i,j = 1,2,3)$, as shown in Appendix A, are the differential operators of u, v and w and the coefficients related to these differential operators are functions of the coordinate variable x.

For this set of partial differential equations (12)–(14), it is impossible to get directly the numerical solutions by using the trial functions (11). However, when the shell of revolution is a rotating cylindrical shell, it is possible to obtain directly the numerical solutions because the coefficients of the partial differential equations of cylindrical shell are then independent of x, i.e. constants (see Lam *et al.* 1995b).

As an example, consider the coefficient P_{11} related to U in the resulting equation after substituting the trial function (11) into equation (12). For the case of a cylindrical shell, $P_{11} = C_{11}$ is a constant independent of the coordinate variable x, the expression of which is as follows:

$$P_{11} = C_{11} = \frac{L}{2}\left(\omega^2\rho h - \Omega^2\rho h n^2 - \frac{A_{11}m^2\pi^2}{L^2} - \frac{A_{66}n^2}{R^2}\right) \tag{15}$$

where $A_{ii}(i = 1, 6)$ are the tensile stiffness in the constitutive relations of the conical shell and are given in Appendix A. For conical shell, however, $P_{11} = P_{11}(x)$, i.e. a function of the coordinate variable x, the expression of which is written as:

$$P_{11} = P_{11}(x) = \frac{\omega^2\rho h L^2 - A_{11}m^2\pi^2}{L^2}\cos\left(\frac{m\pi x}{L}\right) - \frac{A_{11}m\pi\sin\alpha}{L(a+x\sin\alpha)}\sin\left(\frac{m\pi x}{L}\right)$$
$$-\frac{(A_{66}n^2 + a^2n^2\Omega^2\rho h + A_{22}\sin^2\alpha + 2an^2\Omega^2\rho h x\sin\alpha + n^2\Omega^2\rho h x^2\sin^2\alpha)}{(a+x\sin\alpha)^2}\cos\left(\frac{m\pi x}{L}\right). \tag{16}$$

Thus, for the cylindrical shell, $P_{ij} = C_{ij}(i,j = 1, 2, 3)$ are constants, so we can calculate directly its eigenvalues (see Lam *et al.*, 1995b). But for the conical shell, as $P_{ij} = P_{ij}(x)(i,j = 1, 2, 3)$, we should first use some approximate numerical methods, such as the Galerkin's method which will be used in this paper, in the governing equations so as to obtain the coefficients $C_{ij}(i,j = 1, 2, 3)$ independent of the coordinate variable x. In this way, we can then obtain the eigenvalues discussed in the paper. This is the main difference between the analyses of the free vibrations of a rotating conical shell and a rotating cylindrical shell. This is also the main reason why the analysis of free vibration of a rotating conical shell is much more difficult and complicated than that of a rotating cylindrical shell.

By substituting the trial function (11) into equations (12)–(14), the use of Galerkin's method results in:

$$\int_t\int_\theta\int_x (P_{11}U + P_{12}V + P_{13}W)u\,dx\,d\theta\,dt = 0 \tag{17}$$

$$\int_t\int_\theta\int_x (P_{21}U + P_{22}V + P_{23}W)v\,dx\,d\theta\,dt = 0 \tag{18}$$

$$\int_t\int_\theta\int_x (P_{31}U + P_{32}V + P_{33}W)w\,dx\,d\theta\,dt = 0. \tag{19}$$

After performing the integrations, equations (17)–(19) can be written in the following matrix form:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{20}$$

where the coefficients $C_{ij}(i,j = 1, 2, 3)$ are some very complicated and long expressions in terms of material constants and geometric parameters. C_{11} is the simplest one amongst all the coefficients C_{ij} and is given for information in Appendix B.

For equation (20) to have non-trivial solutions, the determinant of the characteristic matrix in the above equation is set equal to zero,

$$\begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = 0. \tag{21}$$

Expanding equation (21) can yield the following equation,

$$d_0\omega^6 + d_1\omega^5 + d_2\omega^4 + d_3\omega^3 + d_4\omega^2 + d_5\omega + d_6 = 0. \tag{22}$$

The equation (22) has six roots. From our calculating results, it is known that the two roots whose absolute values are the smallest are real numbers, one positive and the other negative. These two eigenvalues of real numbers correspond to the backward and forward travelling waves as well as to the positive and negative rotating speeds of the conical shell, respectively. A detailed discussion of the two eigenvalues of the real numbers will be made in the next section.

NUMERICAL RESULTS AND DISCUSSION

To examine the accuracy of the present work, two comparisons with results in the open literature are made. The first involves free vibration of a rotating cylindrical shell by taking $\alpha = 0$ in the present formulations. The conical shell hence becomes a cylindrical shell. The second involves the free vibration of a non-rotating conical shell by taking $\Omega = 0$ in the present formulations. The results obtained are shown in Tables 1 and 2.

For ease of discussion and comparison with results available in the literature, a frequency parameter f is used; here f is defined as

$$f = \omega b\sqrt{\frac{(1-\mu^2)\rho}{E}}. \tag{23}$$

By examining Tables 1 and 2, it can be seen that the present formulations agree very well with the results available in the literature (Chen *et al.*, 1993; Irie *et al.*, 1984), indicating the accuracy of the present work.

The frequency parameter of the free vibration solution of a rotating conical shell is a function of the rotating speed. At a given rotating speed, the eigensolution for each mode of the vibration, i.e. for each pair of the wave numbers (m, n), where m is the meridional wave number and n is the circumferential wave number, consisted of a positive and a negative eigenvalue mentioned above. These two eigenvalues corresponded to the backward and forward travelling waves or the positive and negative rotating speeds of the conical shell, respectively. The positive eigenvalue corresponded to the backward waves due to a rotation $\Omega > 0$ and the negative eigenvalue corresponded to the forward waves due to a rotation $\Omega < 0$. In the case of a stationary conical shell, these two eigenvalues are identical and the vibrational motion of the conical shell is hence a standing wave motion. However, when the conical shell starts to rotate, the standing wave motion will be transformed and depending on the rotating direction, backward or forward waves will be present. It can be shown that the absolute values of the backward waves are always larger than those of forward waves through the analysis of the present numerical results.

In this paper, studies on the vibration of a truncated circular rotating conical shell are presented. The studies focused on the relationship between the frequency parameter f and the circumferential wave number n at various rotating speeds and cone angles. Results are also presented for the relationship between the frequency parameter f and rotating speed Ω for various cone angles and geometric properties (L/a ratios). The variations of frequency parameter at various modes of free vibration and rotating speed of the truncated cone are also presented.

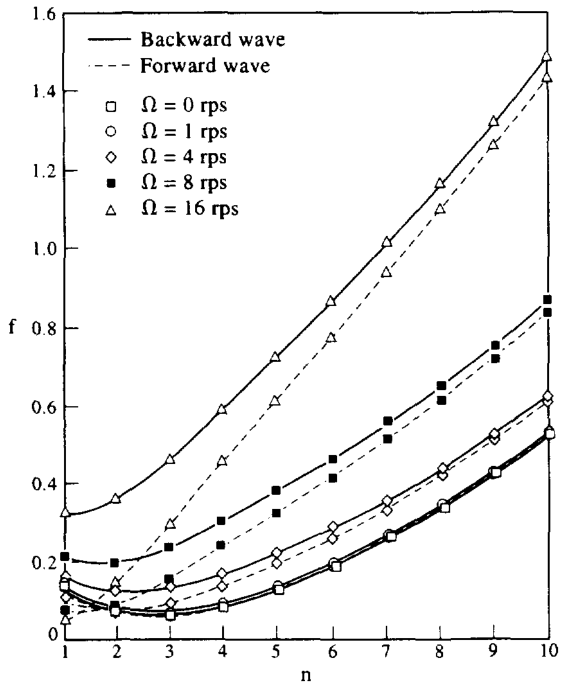

Fig. 2. Relationship between the frequency parameter f and circumferential wave number n at various rotating speeds for cone angle $\alpha = 5^{\circ}$ ($m = 1, \mu = 0.3, h/a = 0.02, L/a = 20$).

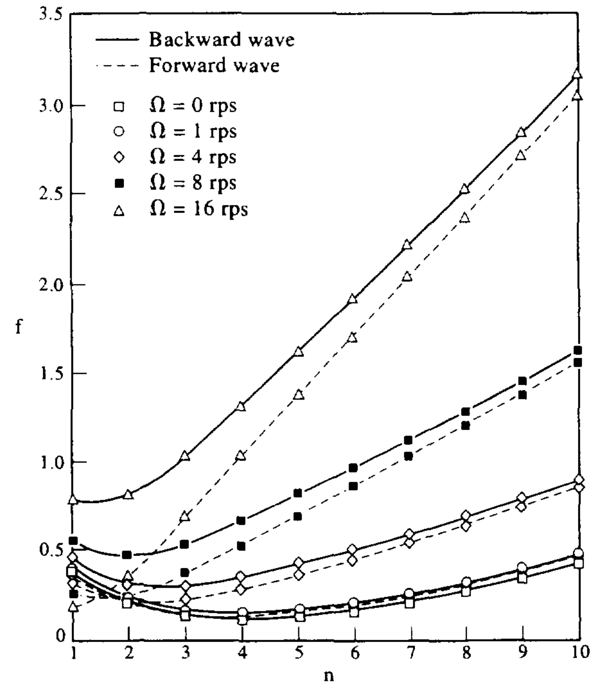

Fig. 3. Relationship between the frequency parameter f and circumferential wave number n at various rotating speeds for cone angle $\alpha = 15^{\circ}$ ($m = 1, \mu = 0.3, h/a = 0.02, L/a = 20$).

In the presentation of results, the backward wave is represented by a solid line and the forward wave by a dashed line; the unit for rotating speed Ω is rps (revolutions per second or Hz).

Figures 2–6 present the frequency characteristics of free vibrations for the truncated circular rotating conical shell with simply-supported boundary conditions at both ends. Results are presented for five different cone angles, namely, $\alpha = 5^{\circ}$, 15°, 30°, 45° and 60°,

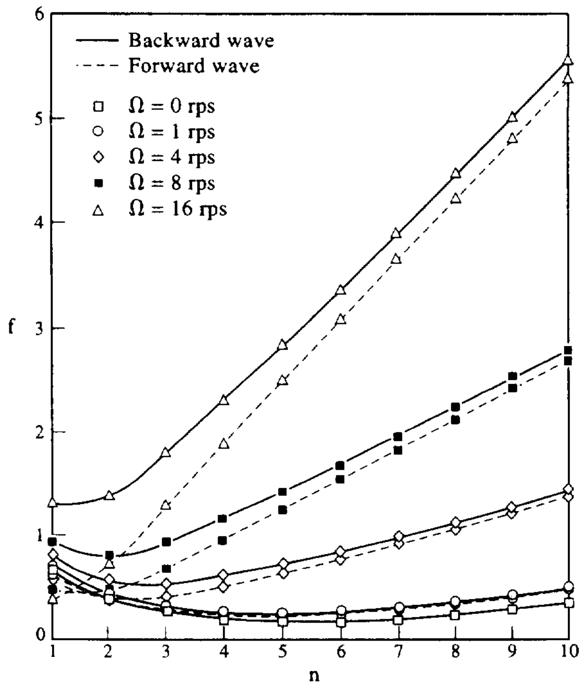

Fig. 4. Relationship between the frequency parameter f and circumferential wave number n at various rotating speeds for cone angle $\alpha = 30$ ($m = 1, \mu = 0.3, h/a = 0.02, L/a = 20$).

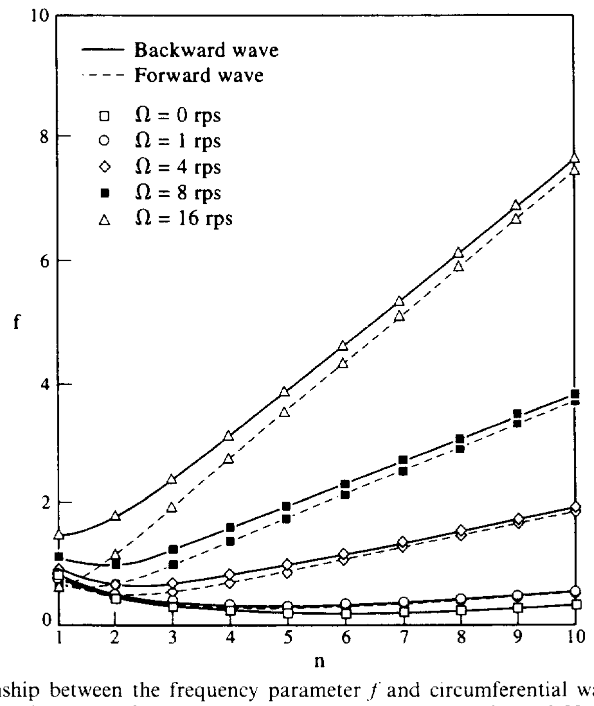

Fig. 5. Relationship between the frequency parameter f and circumferential wave number n at various rotating speeds for cone angle $\alpha = 45$ ($m = 1, \mu = 0.3, h/a = 0.02, L/a = 20$).

at five different rotating speeds, $\Omega = 0, 1, 4, 8$ and 16. From the figures, it can be seen that for the case of a stationary cone, $\Omega = 0$, a standing wave is obtained. For other values of Ω, backward and forward waves are obtained, depending on the sign of the eigenvalue. From the figures, it can also be seen that the frequency parameter f increases with increasing rotating speed at the same circumferential wave number n or with increasing circumferential wave number at the same rotating speed.

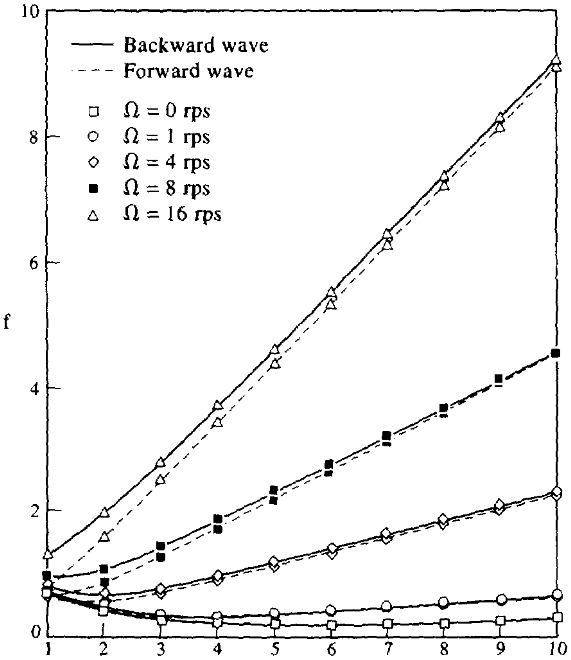

Fig. 6. Relationship between the frequency parameter f and circumferential wave number n at various rotating speeds for cone angle $\alpha = 60$ ($m = 1, \mu = 0.3, h/a = 0.02, L/a = 20$).

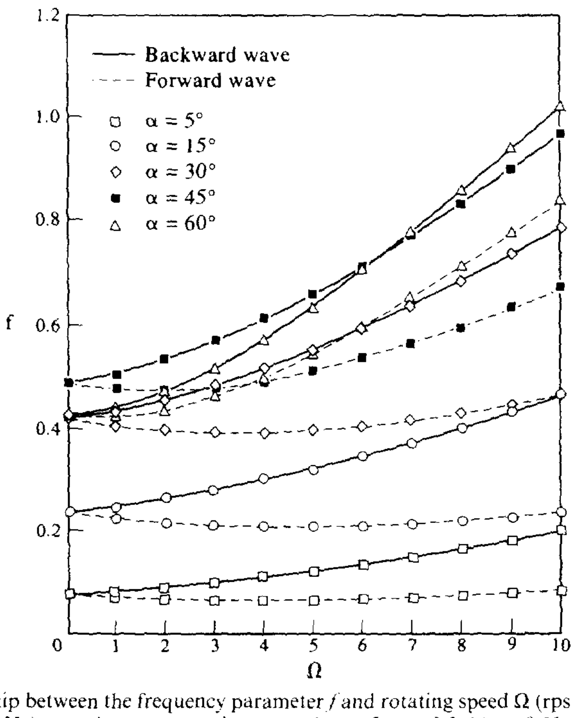

Fig. 7. Relationship between the frequency parameter f and rotating speed Ω (rps = revolutions per second or Hz) at various cone angle α ($m = 1, n = 2, \mu = 0.3, h/a = 0.01, L/a = 15$).

The variations of frequency parameter f against rotating speed Ω for various cone angle α are shown in Fig. 7. The effects of geometric properties are also considered and shown in Fig. 8 and Table 3. In Fig. 8, the variations of frequency parameter against rotating speed for various geometric coefficient L/a are plotted. It can be seen that frequency parameter increases significantly with increasing rotating speed at a fixed L/a; also at a fixed rotating speed, the frequency increases with increasing L/a; hence it can be concluded that the L/a ratios have significant influence on the frequency parameter f.

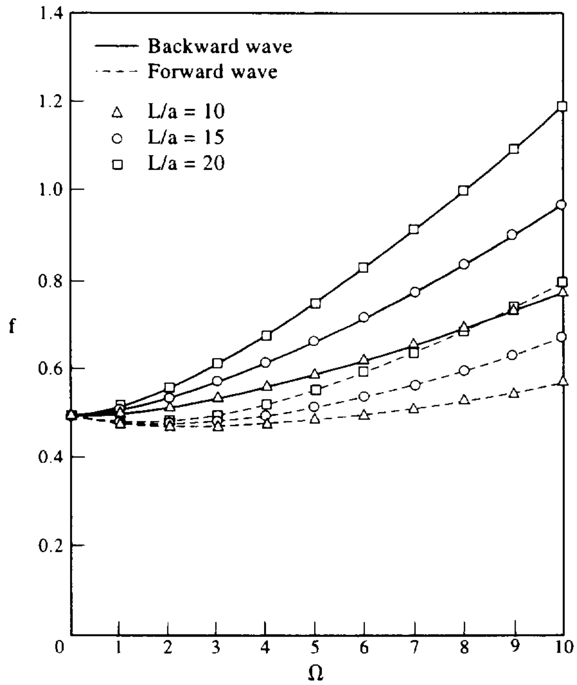

Fig. 8. Relationship between the frequency parameter f and rotating speed Ω (rps = revolutions per second or Hz) at various non-dimensional geometric coefficient L/a ($m = 1, n = 2, \alpha = 45^\circ$, $\mu = 0.3, h/a = 0.02$).

Table 3. Effects of the non-dimensional geometric coefficient h/a of a simply-supported conical shell on the frequency characteristics ($m = 1$, $n = 2$, $\mu = 0.3, \alpha = 45^\circ$, $L/a = 20$)

	$h/a = 0.002$		$h/a = 0.006$	
Ω (rps)	f_b	f_f	f_b	f_f
0	0.49115	0.49115	0.49116	0.49116
1	0.51816	0.47932	0.51817	0.47933
2	0.55968	0.48197	0.55969	0.48198
3	0.61398	0.49735	0.61399	0.49736
4	0.67880	0.52318	0.67881	0.52319
5	0.75194	0.55722	0.75195	0.55723
6	0.83147	0.59753	0.83148	0.59754
7	0.91587	0.64257	0.91588	0.64258
8	1.00397	0.69115	1.00398	0.69116
9	1.09490	0.74236	1.09490	0.74236
10	1.18797	0.79550	1.18797	0.79551
	$h/a = 0.01$		$h/a = 0.02$	
Ω (rps)	f_b	f_f	f_b	f_f
0	0.49119	0.49119	0.49140	0.49140
1	0.51819	0.47935	0.51831	0.47946
2	0.55972	0.48201	0.55983	0.48211
3	0.61401	0.49738	0.61411	0.49748
4	0.67883	0.52321	0.67893	0.52330
5	0.75196	0.55725	0.75205	0.55733
6	0.83149	0.59756	0.83157	0.59764
7	0.91589	0.64260	0.91597	0.64267
8	1.00399	0.69117	1.00406	0.69124
9	1.09492	0.74237	1.09498	0.74243
10	1.18798	0.79552	1.18804	0.79557

However, the same cannot be concluded on the effects of the geometric property h/a. From Table 3, the frequency parameter for both backward and forward waves at various rotating speeds and h/a ratios are tabulated. It can be seen that the effects of the h/a ratios

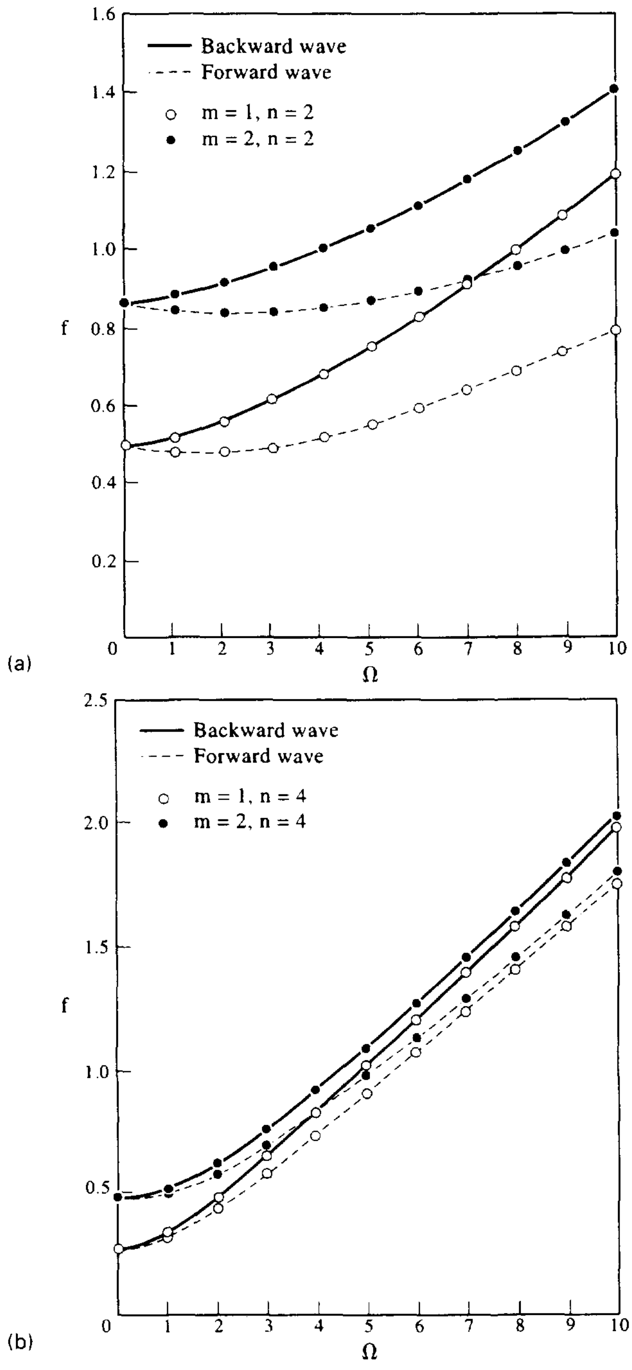

Fig. 9. Relationship between the frequency parameter f at various modes of free vibration and rotating speed Ω ($\mu = 0.3$, $\alpha = 45°$, $h/a = 0.02$, $L/a = 20$).

of the simply-supported truncated circular conical shell on the frequency characteristics are very small when compared with other geometry properties, for example, the cone angle α and the L/a ratio.

The variations of frequency parameter f against rotating speed of the truncated circular conical shell for various modes (m, n) of vibration, where m and n are the meridional and circumferential wave numbers of the vibrations, are shown in Figs 9(a) and (b). The figures show that the mode (2, 2) has higher backward and forward waves frequencies than the mode (1, 2). The same characteristic is also observed between the modes (2, 4) and (1, 4). When the rotating speed became greater, the frequencies for the backward and forward waves at any mode of the vibration (m, n) are observed to increase linearly with the rotational speed.

CONCLUSIONS

A method for studying the vibration of a rotating truncated circular conical shell has been presented. In the present paper, only simply-supported boundary conditions are considered. Parametric studies on the frequency characteristics at different modes of vibration, geometric properties and rotating speeds are presented. The results from the present analysis are also compared with those available in the literatures involving a very long rotating cylindrical shell and a stationary conical shell and very good agreement is obtained.

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APPENDIX A

The differential operators L_{ij} in the equations (12)–(14) are given as follows:

$$L_{11} = -\frac{A_{22}\sin^2\alpha}{r^2(x)} - \rho h\frac{\partial^2}{\partial t^2} + \left(\frac{A_{66}}{r^2(x)} + \Omega^2\rho h\right)\frac{\partial^2}{\partial\theta^2} + \frac{A_{11}\sin\alpha}{r(x)}\frac{\partial}{\partial x} + A_{11}\frac{\partial^2}{\partial x^2} \tag{A1}$$

$$L_{12} = 2\Omega\rho h\sin\alpha\frac{\partial}{\partial t} - \frac{(A_{22}+A_{66})\sin\alpha}{r^2(x)}\frac{\partial}{\partial\theta} + \frac{(A_{12}+A_{66})}{r(x)}\frac{\partial^2}{\partial x\,\partial\theta} \tag{A2}$$

$$L_{13} = -\frac{A_{22}\cos\alpha\sin\alpha}{r^2(x)} + \left(\frac{A_{12}\cos\alpha}{r(x)} - \Omega^2\rho h r(x)\cos\alpha\right)\frac{\partial}{\partial x} \tag{A3}$$

$$L_{21} = -2\Omega\rho h\sin\alpha\frac{\partial}{\partial t} + \left(\frac{(A_{22}+A_{66})\sin\alpha}{r^2(x)} + \Omega^2\rho h\sin\alpha\right)\frac{\partial}{\partial\theta} + \left(\Omega^2\rho h r(x) + \frac{(A_{21}+A_{66})}{r(x)}\right)\frac{\partial^2}{\partial x\,\partial\theta} \tag{A4}$$

$$L_{22} = \frac{4D_{66}\cos^2\alpha\sin^2\alpha}{r^4(x)} - \frac{A_{66}\sin^2\alpha}{r^2(x)} - \rho h\frac{\partial^2}{\partial t^2} + \left(\frac{D_{22}\cos^2\alpha}{r^4(x)} + \frac{A_{22}}{r^2(x)}\right)\frac{\partial^2}{\partial\theta^2}$$

$$-\left(\frac{4D_{66}\cos^2\alpha\sin\alpha}{r^3(x)} - \frac{A_{66}\sin\alpha}{r(x)} - \Omega^2\rho h r(x)\sin\alpha\right)\frac{\partial}{\partial x} + \left(\frac{2D_{66}\cos^2\alpha}{r^2(x)} + A_{66}\right)\frac{\partial^2}{\partial x^2} \tag{A5}$$

$$L_{23} = -2\Omega\rho h\cos\alpha\frac{\partial}{\partial t} - \left(\frac{4D_{66}\cos\alpha\sin^2\alpha}{r^4(x)} - \frac{A_{22}\cos\alpha}{r^2(x)}\right)\frac{\partial}{\partial\theta} - \frac{D_{22}\cos\alpha}{r^4(x)}\frac{\partial^3}{\partial\theta^3}$$

$$-\frac{(D_{22}-4D_{66})\cos\alpha\sin\alpha}{r^3(x)}\frac{\partial^2}{\partial x\,\partial\theta} - \frac{(D_{21}+2D_{66})\cos\alpha}{r^2(x)}\frac{\partial^3}{\partial x^2\,\partial\theta} \tag{A6}$$

$$L_{31} = -\frac{A_{22}\cos\alpha\sin\alpha}{r^2(x)} + \Omega^2\rho h\cos\alpha\sin\alpha - \left(\frac{A_{21}\cos\alpha}{r(x)} + \Omega^2\rho h r(x)\cos\alpha\right)\frac{\partial}{\partial x} \tag{A7}$$

$$L_{32} = 2\Omega\rho h\cos\alpha\frac{\partial}{\partial t} + \frac{D_{22}\cos\alpha}{r^4(x)}\frac{\partial^3}{\partial\theta^3}$$

$$+\left(\frac{2(D_{12}+D_{22}+4D_{66})\cos\alpha\sin^2\alpha}{r^4(x)} - \frac{A_{22}\cos\alpha}{r^2(x)}\right)\frac{\partial}{\partial\theta}$$

$$-\frac{(2D_{12}+D_{22}+8D_{66})\cos\alpha\sin\alpha}{r^3(x)}\frac{\partial^2}{\partial x\,\partial\theta} + \frac{(D_{12}+4D_{66})\cos\alpha}{r^2(x)}\frac{\partial^3}{\partial x^2\,\partial\theta} \tag{A8}$$

$$L_{33} = -\frac{A_{22}\cos^2\alpha}{r^2(x)} + \Omega^2\rho h\cos^2\alpha - \rho h\frac{\partial^2}{\partial t^2} - \frac{D_{22}}{r^4(x)}\frac{\partial^4}{\partial\theta^4}$$

$$-\left(\frac{2(D_{12}+D_{22}+4D_{66})\sin^2\alpha}{r^4(x)} + \Omega^2\rho h\right)\frac{\partial^2}{\partial\theta^2} - \frac{D_{22}\sin^3\alpha}{r^3(x)}\frac{\partial}{\partial x}$$

$$+\frac{2(D_{12}+4D_{66})\sin\alpha}{r^3(x)}\frac{\partial^3}{\partial x\,\partial\theta^2} + \frac{D_{22}\sin^2\alpha}{r^2(x)}\frac{\partial^2}{\partial x^2}$$

$$-\frac{(D_{12}+D_{21}+4D_{66})}{r^2(x)}\frac{\partial^4}{\partial x^2\,\partial\theta^2} - \frac{2D_{11}\sin\alpha}{r(x)}\frac{\partial^3}{\partial x^3} - D_{11}\frac{\partial^4}{\partial x^4} \tag{A9}$$

here, A_{ij} and D_{ij} are the tensile and bending stiffnesses, respectively, in the constitutive equations (9). For the present isotropic conical shell,

$$A_{11} = A_{22} = \frac{Eh}{(1-\mu^2)} \quad A_{12} = A_{21} = \frac{\mu Eh}{(1-\mu^2)} \quad A_{66} = \frac{Eh}{2(1+\mu)}$$

$$D_{11} = D_{22} = \frac{Eh^3}{12(1-\mu^2)} \quad D_{12} = D_{21} = \frac{\mu Eh^3}{12(1-\mu^2)} \quad D_{66} = \frac{Eh^3}{12(1+\mu)}. \tag{A10}$$

APPENDIX B

C_{11}, which is the simplest expression amongst all the coefficients C_{ij} in the equation (20), is given here for information:

$$C_{11} = \frac{L\rho h}{2}(\omega^2-\Omega^2 n^2) - \frac{1}{2L}(A_{11}m^2\pi^2) - \frac{1}{a}(A_{66}n^2\csc\alpha + A_{22}\sin\alpha)$$

$$-\frac{1}{b}A_{22}m\pi\cos(\alpha-(2am\pi\csc\alpha)/L)Ci(2m\pi+(2am\pi\csc\alpha)/L)$$

$$+\frac{1}{b}A_{22}m\pi\cos(\alpha+(2am\pi\csc\alpha)/L)Ci(2m\pi+(2am\pi\csc\alpha)/L)$$

$$-\frac{1}{2b}A_{66}mn^2\pi\cos(\alpha-(2am\pi\csc\alpha)/L)Ci(2m\pi+(2am\pi\csc\alpha)/L)\csc^2\alpha$$

$$+\frac{1}{2b}A_{66}mn^2\pi\cos(\alpha+(2am\pi\csc\alpha)/L)Ci(2m\pi+(2am\pi\csc\alpha)/L)\csc^2\alpha$$

$$+\frac{A_{22}\sin\alpha}{b}+\frac{A_{66}n^2\csc^2\alpha\sin\alpha}{2b}+\frac{1}{2b}(A_{66}n^2\csc\alpha)$$

$$-\frac{1}{2L}A_{11}m\pi Ci(2am\pi\csc\alpha/L)\sin(2am\pi\csc\alpha/L)$$

$$+\frac{1}{L}A_{22}m\pi Ci(2am\pi\csc\alpha/L)\sin(2am\pi\csc\alpha/L)$$

$$+\frac{1}{2L}A_{11}m\pi Ci(2m\pi+(2am\pi\csc\alpha)/L)\sin(2am\pi\csc\alpha/L)$$

$$+\frac{1}{L}(A_{66}mn^2\pi Ci(2am\pi\csc\alpha/L)\csc^2\alpha\sin(2am\pi\csc\alpha/L))$$

$$-\frac{a}{Lb}(A_{22}m\pi Ci(2m\pi+(2am\pi\csc\alpha)/L)\sin(2am\pi\csc\alpha/L))$$

$$-\frac{a}{Lb}(A_{66}mn^2\pi Ci(2m\pi+(2am\pi\csc\alpha)/L)\csc^2\alpha\sin(2am\pi\csc\alpha/L))$$

$$+\frac{1}{2L}(A_{11}m\pi\cos(2am\pi\csc\alpha/L)Si(2am\pi\csc\alpha/L))$$

$$-\frac{1}{L}(A_{22}m\pi\cos(2am\pi\csc\alpha/L)Si(2am\pi\csc\alpha/L))$$

$$-\frac{1}{L}(A_{66}mn^2\pi\cos(2am\pi\csc\alpha/L)\csc^2\alpha Si(2am\pi\csc\alpha/L))$$

$$-\frac{1}{2L}(A_{11}m\pi\cos(2am\pi\csc\alpha/L)Si(2m\pi+(2am\pi\csc\alpha)/L))$$

$$+\frac{a}{Lb}(A_{22}m\pi\cos(2am\pi\csc\alpha/L)Si(2m\pi+(2am\pi\csc\alpha)/L))$$

$$-\frac{a}{Lb}(A_{66}mn^2\pi\cos(2am\pi\csc\alpha/L)\csc^2\alpha Si(2m\pi+(2am\pi\csc\alpha)/L))$$

$$+\frac{1}{2b}(A_{22}m\pi\sin(\alpha-(2am\pi\csc\alpha)/L)Si(2m\pi+(2am\pi\csc\alpha)/L))$$

$$+\frac{1}{2b}(A_{66}mn^2\pi\csc^2\alpha\sin(\alpha-(2am\pi\csc\alpha)/L)Si(2m\pi+(2am\pi\csc\alpha)/L))$$

$$+\frac{1}{2b}(A_{22}m\pi\sin(\alpha+(2am\pi\csc\alpha)/L)Si(2m\pi+(2am\pi\csc\alpha)/L))$$

$$-\frac{1}{2b}(A_{66}mn^2\pi\csc^2\alpha\sin(\alpha+(2am\pi\csc\alpha)/L)Si(2m\pi+(2am\pi\csc\alpha)/L)) \quad \text{(B1)}$$

here.

$$Si(x)=\int_0^x\frac{\sin(t)}{t}\,\mathrm{d}t\quad Ci(x)=-\int_x^{\infty}\frac{\cos(t)}{t}\,\mathrm{d}t\quad \csc\alpha=\frac{1}{\sin\alpha}. \quad \text{(B2)}$$

It is clear that C_{ij} for a conical shell is a very complicated expression in terms of material constants and geometric parameters when compared with C_{11} for a cylindrical shell (see Lam *et al.* 1995b), which is:

$$C_{11}=\frac{L}{2}\left(\omega^2\rho h-\Omega^2\rho hn^2-\frac{A_{11}m^2\pi^2}{L^2}-\frac{A_{66}n^2}{R^2}\right). \quad \text{(B3)}$$

The same holds for other $C_{ij}(i,j=1,2,3)$.